

Compactness of Embeddings ^{*†}

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Abstract

An improvement of the author's result, proved in 1961, concerning necessary and sufficient conditions for the compactness of embedding operators is given. A counterexample to a published statement concerning compactness of embedding operators is constructed.

1 Introduction

The basic result of this note is:

Theorem 1. *Let $X_1 \subset X_2 \subset X_3$ be Banach spaces, $\|u\|_1 \geq \|u\|_2 \geq \|u\|_3$ (i.e., the norms are comparable) and if $\|u_n\|_3 \rightarrow 0$ as $n \rightarrow \infty$ and u_n is fundamental in X_2 , then $\|u_n\|_2 \rightarrow 0$, (i.e., the norms in X_2 and X_3 are compatible). Under the above assumptions the embedding operator $i : X_1 \rightarrow X_2$ is compact if and only if the following two conditions are valid:*

a) *The embedding operator $j : X_1 \rightarrow X_3$ is compact, and the following inequality holds:*

b) *$\|u\|_2 \leq s\|u\|_1 + c(s)\|u\|_3$, $\forall u \in X_1$, $\forall s \in (0, 1)$, where $c(s) > 0$ is a constant.*

This result is an improvement of the author's old result, originally proved in 1961 (see [2]), where X_2 was assumed to be a Hilbert space. The proof of Theorem 1 is simpler than the one in [2]. This proof is borrowed from the recent paper [3]. In addition to this proof, we construct a counterexample to a theorem in [1], p.35, where the validity of the inequality b) in Theorem 1 is claimed without the assumption of the compatibility of the norms of X_2 and X_3 . (see Remark 1 at the end of this note). This counterexample is new.

*key words: Banach spaces, compactness, embedding operator

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2 Proof

1. *The sufficiency of conditions a) and b) for compactness of $i : X_1 \rightarrow X_2$*

Assume that a) and b) hold and let us prove the compactness of i . Let $S = \{u : u \in X_1, \|u\|_1 = 1\}$ be the unit sphere in X_1 . Using assumption a), select a sequence u_n which converges in X_3 . We claim that this sequence converges also in X_2 . Indeed, since $\|u_n\|_1 = 1$, one uses assumption b) to get

$$\|u_n - u_m\|_2 \leq s\|u_n - u_m\|_1 + c(s)\|u_n - u_m\|_3 \leq 2s + c(s)\|u_n - u_m\|_3.$$

Let $\eta > 0$ be an arbitrary small given number. Choose $s > 0$ such that $2s < \frac{1}{2}\eta$, and for a fixed s choose n and m so large that $c(s)\|u_n - u_m\|_3 < \frac{1}{2}\eta$. This is possible because the sequence u_n converges in X_3 . Consequently, $\|u_n - u_m\|_2 \leq \eta$ if n and m are sufficiently large. This means that the sequence u_n converges in X_2 . Thus, the embedding $i : X_1 \rightarrow X_2$ is compact. In the above argument, i.e., in the proof of sufficiency, the compatibility of the norms was not used.

2. *The necessity of the compactness of $i : X_1 \rightarrow X_2$ for conditions a) and b) to hold.*

Assume now that i is compact. Let us prove that conditions a) and b) hold. In the proof of the necessity of these conditions the assumption about the compatibility of the norms of X_2 and X_3 is used essentially. Without this assumption one cannot prove that conditions a) and b) hold. This is proved in the **Remark 1** after the end of the proof of Theorem 1.

If i is compact, then assumption a) holds because $\|u\|_2 \geq \|u\|_3$. Suppose that assumption b) fails. Then there is a sequence u_n and a number $s_0 > 0$ such that $\|u_n\|_1 = 1$ and

$$\|u_n\|_2 \geq s_0 + n\|u_n\|_3. \quad (1)$$

If the embedding operator i is compact and $\|u_n\|_1 = 1$, then one may assume that the sequence u_n converges in X_2 . Its limit cannot be equal to zero, because, by (1), $\|u_n\|_2 \geq s_0 > 0$. The sequence u_n converges in X_3 because $\|u_n - u_m\|_2 \geq \|u_n - u_m\|_3$, and its limit in X_3 is not zero, because the norms in X_3 and in X_2 are compatible. Thus, $\lim_{n \rightarrow \infty} \|u_n\|_3 > 0$.

Thus, (1) implies $\|u_n\|_3 = O(\frac{1}{n}) \rightarrow 0$ as $n \rightarrow \infty$, while $\lim_{n \rightarrow \infty} \|u_n\|_3 > 0$. This is a contradiction, which proves that b) holds.

Theorem 1 is proved. \square

Remark 1. In [1], p. 35, under the name Lions' lemma, the following claim is stated:

Claim ([1], p.35): *Let $X_1 \subset X_2 \subset X_3$ be three Banach spaces. Suppose the embedding $X_1 \rightarrow X_2$ is compact. Then given any $\epsilon > 0$, there is a $K(\epsilon) > 0$, such that $\|u\|_2 \leq \epsilon\|u\|_1 + K(\epsilon)\|u\|_3$ for all $u \in X_1$.*

This claim, is *not correct* because there is no assumption about compatibility of the norms of X_2 and X_3 .

For example, let $L^2(0, 1)$ be the usual Lebesgue space of square integrable functions, $X_3 = L^2(0, 1)$, and X_2 be a Banach space of $L^2(0, 1)$ functions with a finite value at a

fixed point $y \in [0, 1]$ and with the norm

$$\|u\|_2 := \|u\|_{L^2(0,1)} + |u(y)| = \|u\|_3 + |u(y)|.$$

The space X_2 is complete because X_3 is complete and the one-dimensional space, consisting of numbers $u(y)$ with the usual norm $|u(y)|$, is complete. A function $u_0(x) = 0$ for $x \neq 0$ and $u_0(y) = 1$ has the properties

$$\|u_0\|_3 = 0, \quad \|u_0\|_2 = 1.$$

Clearly, $X_2 \subset X_3$, and the norms in X_2 and X_3 are *comparable*, i.e., $\|u\|_3 \leq \|u\|_2$. However, these norms are *not compatible*: there is a convergent to zero sequence $\lim_{n \rightarrow \infty} u_n = 0$ in X_3 such that it does not converge to zero in X_2 , for example, $\lim_{n \rightarrow \infty} \|u_n\|_2 = 1$ in X_2 . For instance, one may take $u_n(x) = u_0(x)$ for all $n = 1, 2, \dots$, and an arbitrary fixed $y \in [0, 1]$. Then $\|u_n\|_2 = 1$ and $\|u_n\|_3 = 0$, $\lim_{n \rightarrow \infty} \|u_n\|_2 = 1$ and $\lim_{n \rightarrow \infty} \|u_n\|_3 = 0$. The sequence u_n converges to zero in X_3 and to a non-zero element u_0 in X_2 . In this case inequality (1) holds for any fixed $s_0 \in (0, 1)$ and any n , but the contradiction, which was used in the proof of the necessity in Theorem 1, can not be obtained because $\|u_n\|_3 = 0$ for all n .

Let us construct a counterexample which shows that the Claim in [1], mentioned above, is not correct. Fix a $y \in [0, 1]$. Choose the one-dimensional space of functions $\{u : u = \lambda u_0(x)\}$ as X_1 , where $\lambda = \text{const}$, and define the norm in X_1 by the formula $\|u\|_1 = |\lambda|$. Let $X_3 = L^2(0, 1)$. The space X_1 is a one-dimensional Banach space. Therefore bounded sets in X_1 are precompact. Note that $|\lambda| = \|\lambda u_0\|_1 = \|\lambda u_0\|_2 \geq \|\lambda u_0\|_3 = 0$ because $\|u_0\|_3 = 0$. Here the Banach space X_2 is defined as above with the norm $\|u\|_2 := \|u\|_{L^2(0,1)} + |u(y)|$, and the equalities $\|u_0\|_2 = 1$ and $\|u_0\|_3 = 0$ are used.

Consequently,

$$X_1 \subset X_2 \subset X_3, \quad \|u\|_1 \geq \|u\|_2 \geq \|u\|_3,$$

and the embedding $i : X_1 \rightarrow X_2$ is compact because bounded sets in finite-dimensional spaces are precompact and X_1 is a one-dimensional space. Thus, all the assumptions of the **Claim** are satisfied. However the inequality of the **Claim**:

$$\|u\|_2 \leq \epsilon \|u\|_1 + K(\epsilon) \|u\|_3 \quad \forall u \in X_1$$

does not hold for any fixed $\epsilon \in (0, 1)$. In our counterexample $u = \lambda u_0$, $\|u_0\|_3 = 0$, and the above inequality takes the form: $|\lambda| \leq \epsilon |\lambda|$. Clearly, this inequality does not hold for a fixed $\epsilon \in (0, 1)$ unless $\lambda = 0$.

References

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